

Ground state energy and mass gap of a generalized quantum spin ladder

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2000 J. Phys. A: Math. Gen. 33 443

(<http://iopscience.iop.org/0305-4470/33/3/301>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.123

The article was downloaded on 02/06/2010 at 08:28

Please note that [terms and conditions apply](#).

Ground state energy and mass gap of a generalized quantum spin ladder

M T Batchelor and M Maslen

Department of Mathematics, School of Mathematical Sciences, The Australian National University, Canberra ACT 0200, Australia

Received 4 August 1999

Abstract. We show that a two-leg ladder Hamiltonian introduced recently by Albeverio and Fei can be made to satisfy the Hecke algebra. As a result we have found an equivalent representation of the eigenspectrum in terms of the spin- $\frac{1}{2}$ antiferromagnetic XXZ chain at $\Delta = -\frac{5}{3}$. The values of thermodynamic quantities such as the ground state energy and mass gap follow from the known XXZ results.

1. Introduction

A number of solvable models of quantum spin ladders have been found recently. In some models certain combinations of couplings exist such that ground states can be constructed in simple form [1]. Other models have an underlying R -matrix and are thus amenable to the machinery of exactly solvable lattice models in statistical mechanics. Of particular relevance here are the two-leg Heisenberg ladders of Albeverio and Fei [2] and Wang [3]. The two models discussed by Wang are related to known $su(4)$ and $su(1|3)$ R -matrices, and thus to one-dimensional chains. These symmetries are broken by the inclusion of rung interactions of arbitrary strength which preserve the integrability. The rung interactions appear as chemical potentials in the equivalent one-dimensional chains, with only a slight modification to their Bethe ansatz solution. On the other hand, the model introduced by Albeverio and Fei has rung interactions already included in the given interactions around an elementary face of the ladder. The model has not been solved in general. Those authors found the eigenvalues in the sector of the Hamiltonian in which one spin is flipped from the ferromagnetic ground state. Here we address how this model relates to other known models and deduce some exact results from this equivalence.

For simplicity, we have considered open boundary conditions. Here $S = \frac{1}{2}(\sigma^x, \sigma^y, \sigma^z)$ and $T = \frac{1}{2}(\sigma^x, \sigma^y, \sigma^z)$ are the usual spin- $\frac{1}{2}$ operators, with S_i and T_i spin operators on the i th rung of each leg of the ladder. The ladder has L rungs.

2. Hecke algebra

Our starting point is to note that the operator h_j obeys an algebra. Specifically, if we define

$$U_j = \frac{8}{3}(h_j + \frac{1}{8}) \tag{6}$$

then U_j obeys the well known Hecke algebra, which we write here as

$$U_j^2 = (q + q^{-1}) U_j \tag{7}$$

$$U_j U_{j+1} U_j - U_j = U_{j+1} U_j U_{j+1} - U_{j+1} \tag{8}$$

$$[U_i, U_j] = 0 \quad \text{for } |i - j| > 1 \tag{9}$$

with $q + q^{-1} = \frac{10}{3}$. Thus $q = \frac{1}{3}$ or 3.

A number of models satisfy the Hecke algebra [4]. The 4×4 representation of interest here is [5]

$$U_j = \begin{pmatrix} q + q^{-1} & & & \\ & q & 1 & \\ & 1 & q^{-1} & \\ & & & q + q^{-1} \end{pmatrix}. \tag{10}$$

This is the co-product of the Casimir element belonging to the centre of $U_q(su(2))$. The representation (10) is to be compared with the more well known representation

$$U_j = \begin{pmatrix} 0 & & & \\ & q & 1 & \\ & 1 & q^{-1} & \\ & & & 0 \end{pmatrix} \tag{11}$$

of the Temperley–Lieb algebra. The latter satisfies the relations (7)–(9), but with $U_j U_{j\pm 1} U_j - U_j = 0$, and is thus a Hecke quotient. The representation (10) may be written in terms of spin- $\frac{1}{2}$ operators as

$$U_j = \frac{1}{2}(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + \frac{1}{4}(q + q^{-1})(\sigma_j^z \sigma_{j+1}^z - \mathbf{1}) + \frac{1}{4}(q^{-1} - q)(\sigma_{j+1}^z - \sigma_j^z) + (q + q^{-1})\mathbf{1}. \tag{12}$$

It follows that the Hecke Hamiltonian made up of spin- $\frac{1}{2}$ operators can be written as

$$H_{\text{Hecke}} = \sum_{j=1}^{L-1} U_j = \frac{3}{4}(q + q^{-1})(L - 1) + \frac{1}{2} \sum_{j=1}^{L-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \Delta \sigma_j^z \sigma_{j+1}^z) + \frac{1}{4}(q^{-1} - q)(\sigma_L^z - \sigma_1^z) \tag{13}$$

where

$$\Delta = -\frac{1}{2}(q + q^{-1}). \tag{14}$$

However, writing the XXZ term as $H(\Delta)$, the eigenspectrum of (13) is invariant under the transformation $H(\Delta) = -H(-\Delta)$ and the interchange $q \leftrightarrow q^{-1}$. This gives the eigenvalue equivalence

$$E_{\text{Hecke}} \Leftrightarrow E_{\text{XXZ}} + \frac{3}{4}(q + q^{-1})(L - 1) \tag{15}$$

in which the XXZ Hamiltonian is defined as

$$H_{XXZ} = -\frac{1}{2} \sum_{j=1}^{L-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z) + \frac{1}{2} p (\sigma_L^z - \sigma_1^z) \quad (16)$$

with boundary fields

$$p = \frac{1}{2} (q^{-1} - q). \quad (17)$$

This latter Hamiltonian is precisely that of the open antiferromagnetic spin- $\frac{1}{2}$ XXZ chain with fields $\pm p$ at the ends of the chain. Recalling that $q = \frac{1}{3}$, the eigenspectrum of the open spin ladder Hamiltonian (5) is thus equivalent to that of the open XXZ chain with $\Delta = -\frac{5}{3}$ and boundary fields $\pm \frac{2}{3}$, after an appropriate rescaling through equation (6). In particular, for $J > 0$ the Hamiltonian (5) is equivalent to the antiferromagnetic XXZ chain, whilst for $J < 0$ the equivalence is with the ferromagnetic XXZ chain. For simplicity we take $J = \pm 1$.

The eigenvalue equivalence (15) assumes that the two representations of the Hecke algebra are faithful, i.e. although the representations differ in size, they share all eigenvalues in common. Only the multiplicity of eigenvalues differ. For a given number of rungs L , the ladder Hamiltonian (5) is of size $16^{L-1} \times 16^{L-1}$, whilst the equivalent XXZ Hamiltonian is of size $4^{L-1} \times 4^{L-1}$. We have compared the eigenspectrum of each Hamiltonian with increasing L and believe that the representations are indeed faithful. In fact, the entire situation is somewhat analogous to the history of the spin-1 biquadratic chain. That model [6] was mapped to the XXZ chain via the Temperley–Lieb algebra, from which quantities such as the mass gap and the ground state energy, etc were obtained [7]. These were seen to be in agreement with exact inversion relation calculations on the model itself [8]. The spin-1 biquadratic chain was later solved via the Bethe ansatz [9].

From (6) we have

$$H_{\text{ladder}} = \frac{3}{8} H_{\text{Hecke}} - \frac{1}{8} (L - 1). \quad (18)$$

On the other hand, from (15) we have the eigenvalue equivalence $E_{\text{Hecke}} \Leftrightarrow E_{XXZ} + \frac{5}{2} (L - 1)$ and thus

$$E_{\text{ladder}} \Leftrightarrow \frac{3}{8} E_{XXZ} + \frac{13}{16} (L - 1). \quad (19)$$

This is our key result.

3. Ground state energy and mass gap

The open XXZ chain with arbitrary boundary fields has been solved by means of the Bethe ansatz [10]. In particular, the solution for the case $\Delta^2 - p^2 = 1$, as applies here, simplifies considerably. We shall not reproduce the equations here, but content ourselves with recalling the relevant results. Consider the antiferromagnetic case first. In the massive region $\Delta < -1$ it is convenient to define $q = e^{-\theta}$. Here the ground state energy per site, the surface free energy and the mass gap have all been derived [11]†. For the given XXZ normalization, the mass gap is

$$\begin{aligned} \Lambda_{XXZ} &= 2 \sinh \theta \prod_{n=1}^{\infty} \left(\frac{1 - q^n}{1 + q^n} \right)^2 \\ &= \frac{8}{3} \prod_{n=1}^{\infty} \left(\frac{3^n - 1}{3^n + 1} \right)^2 = \frac{8}{3} 0.128 108 \dots \end{aligned} \quad (20)$$

† Of course, the expressions for the ground state energy and the mass gap are in agreement with those obtained originally [12, 13] for periodic boundary conditions.

It thus follows from (19) that the ladder Hamiltonian (5) has a gap $\Lambda_{\text{ladder}} = 0.128\,108\dots$. More generally we expect all massive excitations in the ladder eigenspectrum to be multiples of this elementary gap.

On the other hand, the ground state energy of the open XXZ chain scales for large N as

$$E_{XXZ} \sim N e_{XXZ} + f_{XXZ}. \quad (21)$$

The surface free energy contribution is given by $f_{XXZ} = g - \frac{1}{4}\Lambda_{XXZ}$, where g is a known, though complicated, expression [11]. The ground state energy per site is given by

$$e_{XXZ} = \frac{1}{2} \cosh \theta - \sinh \theta \left(1 + 4 \sum_{n=1}^{\infty} \frac{1}{1 + e^{2n\theta}} \right). \quad (22)$$

It follows from (19) that the ground state energy per site of the ladder is given by

$$\begin{aligned} e_{\text{ladder}} &= \frac{5}{8} - 2 \sum_{n=1}^{\infty} \frac{1}{1 + 9^n} \\ &= 0.397\,527\dots \end{aligned} \quad (23)$$

The surface free energy relation is $f_{\text{ladder}} = \frac{3}{8}f_{XXZ} - \frac{13}{16}$.

In the ferromagnetic regime $J < 0$ the ground state energy of the ladder corresponds to the trivial ferromagnetic ground state $\frac{1}{4}(q + q^{-1})(L - 1)$ of the open XXZ chain. It follows from (19) that the ground state energy of the ferromagnetic ladder is given by $E_{\text{ladder}} = -\frac{9}{8}(L - 1)$. This value is in agreement with the observation of Albeverio and Fei [2].

4. Discussion

We have shown that a two-leg ladder Hamiltonian introduced recently by Albeverio and Fei [2] can be made to satisfy the Hecke algebra for $q = \frac{1}{3}$. As a result we have found an equivalent representation of the eigenspectrum in terms of the spin- $\frac{1}{2}$ XXZ chain at $\Delta = -\frac{5}{3}$. We considered open boundary conditions for which the equivalent chain has surface fields $\pm\frac{2}{3}$ at the ends of the chain. The values of thermodynamic quantities such as the mass gap (20) and ground state energy per site (23) followed from the known XXZ results. Periodic boundary conditions can also be considered by imposing a twist on the periodic XXZ chain, as was done, for example, for the XXZ chain equivalent to the spin-1 biquadratic model [7].

Albeverio and Fei have noted that like the well known spin-1 Affleck–Kennedy–Lieb–Tasaki chain [14] the two-leg ladder Hamiltonian has no free parameter. However, a free parameter has been introduced into the AKLT model via q -deformation [15, 16]. A form of q -deformation should also exist for the two-leg ladder, corresponding to the variable q in the XXZ chain, again equivalent via the Hecke algebra. However, the precise form of the two-leg Hamiltonian would be very complicated. Nevertheless, a phase transition should exist at which the model becomes massless at the critical value $q = 1$.

Another related point is that just as Wang's two-leg ladders can be generalized to n -leg ladders [17], we might ask whether there is another faithful representation of Hecke, this time of size $64^{L-1} \times 64^{L-1}$, corresponding to a three-leg ladder. Again the Hamiltonian would most likely include all possible interactions.

Although we have seen that the two-leg ladder Hamiltonian provides a representation of the Hecke algebra, the equivalence ultimately lies with the XXZ chain, and thus with the Temperley–Lieb algebra. We thus expect that the Hecke representation we have found here is also a quotient of Hecke. On another tack, considering instead the Temperley–Lieb representation (11) with $q = \frac{1}{3}$, we observe that as is to be expected only part of

the eigenspectrum of the two-leg ladder is recovered. Similarly, if we use Saleur's Hecke representation [18]

$$U_j = \begin{pmatrix} 0 & & & & \\ & q & 1 & & \\ & 1 & q^{-1} & & \\ & & & & \\ & & & & q + q^{-1} \end{pmatrix} \quad (24)$$

with $q = \frac{1}{3}$ we recover part of the ladder eigenspectrum. The latter is known to be a quotient of Hecke [19], but more importantly it is free-fermionic, being equivalent to an XX chain [18]. This gives the free-fermionic part of the ladder eigenspectrum. As a point of further interest Saleur's representation can also be used to give the free-fermionic part of the eigenspectrum of the XXZ chain.

Finally, we note that although the two-leg ladder (5) includes complicated interactions it is nevertheless a model for which exact results can be obtained. It thus provides a useful testbed for numerical calculations on more realistic models.

Acknowledgments

It is a pleasure to thank Jon Links for some helpful remarks. This work has been supported by the Australian Research Council.

Note added. After submitting this paper we were informed by Y Wang that the ladder Hamiltonian has also been mapped to the XXZ chain, using a similar method to that proposed by Wang [3]. That mapping appears in the published version [20] of [2].

References

- [1] See, for example, Kolezhuk A K and Mikeska H-J 1998 *Int. J. Mod. Phys. B* **12** 2325 and references therein
- [2] Albeverio A and Fei S-M 1998 Exactly solvable models of generalized spin ladders *Preprint cond-mat/9807341*
- [3] Wang Y 1999 *Phys. Rev. B* **60** 9236
- [4] Alcaraz F C, Koberle R and Lima-Santos A 1992 *Int. J. Mod. Phys. A* **7** 7615
- [5] Saleur H and Zuber J-B 1991 *String Theory and Quantum Gravity* ed M Green *et al* (Singapore: World Scientific) p 1
- [6] Parkinson J B 1988 *J. Phys. C: Solid State Phys.* **21** 3793
Parkinson J B 1988 *J. Physique C* **8** 1413
- [7] Barber M N and Batchelor M T 1989 *Phys. Rev. B* **40** 4621
- [8] Klümper A 1989 *Europhys. Lett.* **9** 815
Klümper A 1990 *J. Phys. A: Math. Gen.* **23** 809
- [9] Koberle R and Lima-Santos A 1994 *J. Phys. A: Math. Gen.* **27** 5409
- [10] Alcaraz F C, Barber M N, Batchelor M T, Baxter R J and Quispel G R W 1987 *J. Phys. A: Math. Gen.* **20** 6397
- [11] Batchelor M T and Hamer C J 1990 *J. Phys. A: Math. Gen.* **23** 761
- [12] Orbach R 1958 *Phys. Rev.* **112** 309
Walker L R 1959 *Phys. Rev.* **116** 1089
- [13] des Cloizeaux J and Gaudin M 1966 *J. Math. Phys.* **7** 1384
- [14] Affleck I, Kennedy T, Lieb E H and Tasaki H 1988 *Commun. Math. Phys.* **115** 477
- [15] Batchelor M T, Mezincescu L, Nepomechie R I and Rittenberg V 1990 *J. Phys. A: Math. Gen.* **23** L141
Batchelor M T and Yung C M 1994 *Int. J. Mod. Phys. B* **8** 3645
- [16] Koo W M and Saleur H 1993 *Int. J. Mod. Phys. A* **8** 5165
- [17] Batchelor M T and Maslen M 1999 *J. Phys. A: Math. Gen.* **32** L377
- [18] Saleur H 1990 *Trieste Conf. on Recent Developments in Conformal Field Theories* ed S Randjbar-Daemi *et al* (Singapore: World Scientific) p 160
- [19] Martin P P and Rittenberg V 1992 *Int. J. Mod. Phys. A* **7** (Suppl. 1B) 707
- [20] Albeverio A, Fei S-M and Wang Y 1999 *Europhys. Lett.* **47** 364